Comparison of Numerical Methods for Nano Boundary Layer Flow

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The nonlinear differential equations describing the nano boundary layer flow is investigated in this paper utilizing Chebyshev collocation scheme. The results obtained in this research are compared with those obtained by the other published works.

Key words: Chebyshev Collocation Method; Nano Boundary Layer.

1. Introduction

The notion of a boundary layer was first introduced by Prandtl [1] over a hundred years ago to explain the discrepancies between the theory of inviscid fluid flow and experiment. In classical boundary layer theory, the condition of no-slip near the solid walls is usually applied. This because the fluid velocity component is assumed to be zero relative to the solid boundary. However, this is not true for fluid flows at the micro and nano scale. Investigations show that the condition of no-slip is no longer valid and instead, a certain degree of tangential slip must be allowed (see [2]). In recent years, some interest has been given to the study of the nano boundary layer flow and some useful results have been introduced by the authors [3 – 9]. In this paper, we consider the model proposed by Wang [7] describing the viscous flow due to a stretching surface with both surface slip and suction (or injection). As in Wang, we consider two geometries situations: (i) the two-dimensional stretching surface and (ii) the axisymmetric stretching surface. A similarity transform is applied in [7] to convert the Navier–Stokes equations into a third-order nonlinear ordinary differential equation given by

$$f'''(\eta) - \left(f'(\eta)\right)^2 + m\, f(\eta)\, f''(\eta) = 0, \qquad (1)$$

where m is a parameter describing the type of stretching. When $m = 1$, we have two-dimensional stretching, while $m = 2$, for axisymmetric stretching [7]. The existence and uniqueness results for each of the two problems were presented in Wang [7] along with some numerical results. The flow is subjected to the following boundary conditions:

$$f(0) = s,\ f'(0) - 1 = K\ f''(0),\ f''(\infty) = 0, \quad (2)$$

where $K > 0$ is a non-dimensional slip parameter and $s < 0$ when injection from the surface occurs and $s > 0$ for suction.

In order to solve the boundary value problem (BVP) given by (1) and (2), various numerical and analytical methods have been proposed. Van Gorder et al. [8] applied the homotopy analysis method to solve the BVP defined above. Also, they discussed the effects of the slip parameter $K > 0$ and the suction parameter $s > 0$ on the fluid velocity and on the tangential stress. As expected, they found that for such fluid flows at nano scales, the shear stress at the wall decreases (in an absolute sense) with an increase in the slip parameter $K > 0$.

The method used by Van Gorder et al. [8] for obtaining numerical solutions differs from that of Wang [7] in that they employed a boundary value problem solver, while Wang [9] converts the boundary value problem into an initial value problem first and then obtains a solution via the Runge–Kutta method. The results obtained by Van Gorder et al. [8] agree with those obtained by Wang [7] up to the number of decimal places provided. For instance, the numerical solutions for the shear stress at the surface $f''(0)$ are given to four decimal places in Wang [7] and to three decimal places in Wang [9]. In addition, Van Gorder et al. [8] considered up to 10 decimal places and the first few digits of their results agree with those of Wang [7, 9].

In this paper, we aim to compare numerically obtained results by using the Chebyshev collocation scheme with those obtained by the other published works.

2. Previous Results

In this section we aim to report some previous results for (1) and (2). At $m = 1$ and $s = K = 0$, Crane [10] gave the exact solution

$$f(\eta) = 1 - \rho^{-\eta}. \tag{3}$$

At arbitrary values of s and K, Wang [7] obtained a solution in the form

$$f(\eta) = s + (C - s)(1 - \rho^{-C\eta}), \tag{4}$$

where C is the positive root of the cubic equation:

$$K\,C^3 + (1 - s\,K)C^2 - s\,C - 1 = 0. \tag{5}$$

When there is no suction, (4) reduces to that of Andersson [11]. Moreover, when there is no slip it reduces to that of Gupta and Gupta [12]. Finally, Crane's solution is recovered when both suction and slip are absent. In the next section we shall introduce the Chebyshev pseudospectral method. We then apply it to solve the BVP given by (1) and (2) in a subsequent section.

3. The Chebyshev Collocation Method

A numerical solution based on Chebyshev collocation approximations seems to be a very good choice in many practical problems (as described in the literature review and for example in Canuto et al. [13] and Peyret [14]). Accordingly, the Chebyshev collocation method will be applied for the presented model. The derivatives of the function $f(x)$ at the Gauss–Lobatto points, $x_k = \cos\left(\frac{k\pi}{L}\right)$, which are the linear combination of the values of the function $f(x)$ [15]

$$f^{(n)} = \mathrm{D}^{(n)} f,$$

where

$$\underline{f} = [f(x_0), f(x_1), \ldots, f(x_L)]^{\mathrm{T}},$$

and

$$\underline{f}^{(n)} = \left[f^{(n)}(x_0), f^{(n)}(x_1), \ldots, f^{(n)}(x_L)\right]^{\mathrm{T}},$$

with

$$\mathrm{D}^{(n)} = \left[\mathrm{d}^{(n)}_{k,j}\right]$$

or

$$f^{(n)}(x_k) = \sum_{j=0}^{L} \mathrm{d}^{(n)}_{k,j} f(x_j),$$

where

$$\mathrm{d}^{(n)}_{k,j} = \frac{2\gamma_j^*}{L} \sum_{l=n}^{L} \sum_{\substack{m=0\\(m+l-n)even}}^{l-n} \gamma_l^* a^n_{m,l}\,(-1)^{\left[\frac{lj}{L}\right]+\left[\frac{mk}{L}\right]}$$

$$x_{lj-L\left[\frac{lj}{L}\right]} x_{mk-L\left[\frac{mk}{L}\right]},$$

with

$$a^n_{m,l} = \frac{2^n l}{(n-1)!c_m} \frac{(s-m+n-1)!\,(s+n-1)!}{(s)!\,(s-m)!},$$

such that $2s = l + m - n$ and $c_0 = 2, c_i = 1, i \geq 1$, where $k,\ j = 0,1,2,\ldots,L$ and $\gamma_0^* = \gamma_l^* = \frac{1}{2}$, $\gamma_j^* = 1$ for $j = 1,2,3,\ldots,L-1$. The round off errors incurred during computing differentiation matrices $\mathrm{D}^{(n)}$ are investigated in [15].

4. Descriptions of the Method for the Governing Equations

In this section the third-order nonlinear ordinary differential equation (1), with boundary conditions (2) are approximated by using the Chebyshev collocation method [15–21]. The grid points (x_i, x_j) in this situation are given as $x_i = \cos\left(\frac{i\pi}{L_1}\right)$, $x_j = \cos\left(\frac{j\pi}{L_2}\right)$ for $i = 1,\ldots,L_1 - 1$ and $j = 1,\ldots,L_2 - 1$. The domain in the x-direction is $[0, x_{\max}]$ where $x_{\max}$ is the length of the dimensionless axial coordinate and the domain in the η-direction is $[0, \eta_{\max}]$ where $\eta_{\max}$ corresponds to η_∞. The domain $[0, x_{\max}] \times [0, \eta_{\max}]$ is mapped into the computational domain $[0, x_{\max}] \times [-1, 1]$ and (1) is transformed into the following equation:

$$\left(\frac{2}{\eta_{\max}}\right)^3 \left(\sum_{l=0}^{L^*} \mathrm{d}^{(3)}_{j,l} f_l\right) - \left(\frac{2}{\eta_{\max}}\right)^2 \left(\sum_{l=0}^{L^*} \mathrm{d}^{(1)}_{j,l} f_l\right)^2$$

$$+ m\left(\frac{2}{\eta_{\max}}\right)^2 f_j \left(\sum_{l=0}^{L^*} \mathrm{d}^{(2)}_{j,l} f_l\right) = 0, \tag{6}$$

satisfying the boundary conditions

$$\begin{cases} f(\eta) = s,\ f'(\eta) - 1 = K f''(\eta) & \text{at } \eta = 0, \\ f'(\eta) = 0, & \text{as } \eta \to \infty. \end{cases} \tag{7}$$

The solution of the above equation (6) with boundary conditions (7), are obtained using the Newton–Raphson iteration technique and these are entered in Table 1 for different values of the governing parameters. The computer program of the numerical method was executed in Mathematica 5.2™ running on a PC.

Table 1. Numerical comparison with results of Van Gorder et al. [8] for the shear stress at the surface $f''(0)$ for $m = 1$ and for various values of k and s.

		$m=1$		$m=2$	
s	k	Van Gorder et al. [8]	Chebyshev collocation method	Van Gorder et al. [8]	Chebyshev collocation method
0	0	−1.0000000000	−1.0002140680	−1.1737207389	−1.1728717518
	0.5	−0.594954851	−0.5949583021	−0.6505276588	−0.6506537515
	1	−0.4301597092	−0.4301515962	−0.4625096440	−0.4624060902
	2	−0.2839796011	−0.2839931755	−0.2990495699	−0.3012992130
	5	−0.1448402050	−0.1449886639	−0.1493933439	−0.1493930000
1	0.5	−0.828471247	−0.8284288652	−1.0696156434	−1.0845357098
	1	−0.5698402910	−0.5698077225	−0.6884145383	−0.6863163502
	2	−0.3558040537	−0.3542856528	−0.4050396378	−0.4050582863
	5	−0.1699818524	−0.1699311235	−0.1823320193	−0.1823323052
2	0	−2.4142135624	−2.2524535869	−4.3424865854	−4.3479838530
	0.5	−1.0508551217	−1.3868739033	−1.3462628561	−1.3464634794
	1	−0.6823278038	−0.8776982062	−0.8028573974	−0.8027553761
	2	−0.4035565857	−0.4737760523	−0.4449439750	−0.444961970
	5	−0.1821716651	−0.1957145791	−0.1905161474	−0.1905160000

Table 2. Numerical comparison with exact sloution of Van Gorder et al. [8].

	$m=1,\ s=k=0$			$m=1,\ s=1,\ k=2$	
η	Van Gorder et al. [8]	Chebyshev collocation method	η	Van Gorder et al. [8]	Chebyshev collocation method
0	0	0	0	1	1
0.002	0.0023951331	0.0023951308	0.014	1.0041963475	1.0042474975
0.009	0.0095232938	0.0095231741	0.057	1.0159529058	1.0161317847
0.021	0.0212154310	0.0212141451	0.123	1.0330764243	1.0332769553
0.037	0.0371984785	0.0371914883	0.20	1.0527495931	1.0523574005
0.058	0.0571076976	0.0570817918	0.3	1.0723100000	1.0699397830
0.11	0.1068857745	0.1067027092	0.39	1.0897600000	1.0834101443

	$m=1,\ s=2,\ k=2$			$m=1,\ s=2,\ k=5$	
	Van Gorder et al. [8]	Chebyshev collocation method		Van Gorder et al. [8]	Chebyshev collocation method
0	2	2	0	2	2
0.002	2.00047082134551	2.00012692817733	0.002	2.0002154300	2.0000513361
0.009	2.00182362782631	2.0004790937997	0.009	2.0008346266	2.0001937634
0.02	2.00389104275034	2.00097896275041	0.02	2.0017815010	2.0003959095
0.03	2.0064289244493	2.0015226460990	0.03	2.0029448510	2.0006157477
0.05	2.009157035108226	2.00200993685037	0.05	2.0041966689	2.0008127555
0.06	2.01179836657788	2.0907962878802	0.06	2.0054099604	2.0009592010

5. Conclusion

The third-order nonlinear boundary value problem describing the nano boundary layer flow has been investigated numerically by using the Chebyshev collocation scheme. It is found in this paper that the numerical results agree better with those obtained by using the homotopy analysis method for $m = 1$ compared to $m = 2$.

[1] L. Prandtl, Über Flüssigkeitsbewegung bei sehr kleiner Reibung, in: Proceedings of the Third International Mathematical Congress (1904).
[2] M. Gad-el-Hak, J. Fluids Eng. **121**, 5 (1999).
[3] M. T. Matthews and J. M. Hill, J. Math. Anal. Appl. **333**, 381 (2006).
[4] M. T. Matthews and J. M. Hill, J. Appl. Math. **72**, 894 (2007).
[5] M. T. Matthews and J. M. Hill, Appl. Math. Lett. A **21**, 810 (2008).
[6] J. Cheng, S. Liao, R. N. Mohapatra, and K. Vajravelu, J. Math. Anal. Appl. **343**, 233 (2008).
[7] C. Y. Wang, Nonlin. Anal.: Real World Appl. **10**, 375 (2009).
[8] R. A. Van Gorder, Erik Sweet, and K. Vajravelu, Commun. Nonlin. Sci. Numer. Simul. **15**, 1494 (2010).
[9] C. Y. Wang, Chem. Eng. Sci. **57**, 3745 (2002).
[10] L. J. Crane, Z. Angew. Math. Phys. **21**, 645 (1970).
[11] H. I. Andersson, Acta Mech. **158**, 121 (2002).
[12] P. S. Gupta and A. S. Gupta, Can. J. Chem. Eng. **55**, 744 (1977).
[13] C. Canuto, M. Y. Hussaini, and T. A. Zang, Spectral Methods in Fluid Dynamics, Springer Verlag, New York, 1988.
[14] R. Peyret, Spectral Methods for Incompressible Viscous Flow, Springer Verlag, New York 2002.
[15] E. M. E. Elbarbary and M. S. El-Sayed, Appl. Numer. Math. **55**, 425 (2005).
[16] E. M. E. Elbarbary and M. El-Kady, Appl. Math. Comput. **139**, 513 (2003).
[17] N. S. Elgazery and N. Y. Abd Elazem, J. Chem. Eng. Commun. **197**, 400 (2010).
[18] N. S. Elgazery and N. Y. Abd Elazem, J. Porous Media **12**, 477 (2009).
[19] N. S. Elgazery and N. Y. Abd Elazem, J. Appl. Appl. Math. **3**, 286 (2008).
[20] N. S. Elgazery and N. Y. Abd Elazem, Int. J. Mech. published online 6 March 2009.
[21] N. S. Elgazery and N. Y. Abd Elazem, Z. Naturforsch. **66a**, 427 (2011).